On the BCSS Proof of the Fundamental Theorem of Algebra

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Abstract

We present a more detailed version of the proof of the Fundamental Theorem of Algebra from Chapter 10 of [BCSS98]. We also discuss some consequences of the BCSS Proof, including a geometric definition of intersection multiplicity, a connection to discriminants, and a homotopy algorithm for finding the roots.

1 Preliminaries

In what follows, we let $x, x_1, \ldots, x_N, c_0, \ldots, c_d$ denote algebraically independent indeterminates. The main theorem we prove is the following:

The Fundamental Theorem of Algebra. *A (univariate) polynomial $f \in \mathbb{C}[x]$ of degree $d \geq 1$ has exactly d roots in $\mathbb{C}$, counting multiplicities.*

Before proceeding, let us fix some notation: We let $f(x) := \gamma_0 + \cdots + \gamma_d x^d \in \mathbb{C}[x]$ be a univariate polynomial of degree d (so $\gamma_d \neq 0$) and observe that, thanks to the last column, the determinant of the matrix

$$\mathrm{Syl}_d(f, f') := \begin{bmatrix} \gamma_0 & \cdots & \gamma_{d-1} & \gamma_d & 0 & \cdots & \cdots & 0 \\ & & & \ddots & & & \ddots & \\ 0 & \cdots & \cdots & 0 & \gamma_0 & \cdots & \gamma_{d-1} & \gamma_d \\ \gamma_1 & \cdots & d\gamma_d & 0 & 0 & \cdots & \cdots & 0 \\ & & & \ddots & & & \ddots & \\ 0 & \cdots & & \cdots & 0 & \gamma_1 & \cdots & d\gamma_d \end{bmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} d-1 \text{ shifting rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} d \text{ shifting rows} \end{array}$$

factors as $\gamma_d \Delta_d(\gamma_0, \ldots, \gamma_d)$ for some polynomial $\Delta_d \in \mathbb{Z}[c_0, \ldots, c_d]$. We call Δ_d the *degree d (univariate) discriminant polynomial*. That Δ_d is in fact not identically zero is a consequence of the following formula.

Proposition 1.1 *For all $d \geq 1$ we have* $\det \mathrm{Syl}_d(x^d - 1, dx^{d-1}) = (-1)^{d-1} d^d$.

Proof: One need only observe that $\mathrm{Syl}_d(x^d - 1, dx^{d-1})$ is upper-triangular, with diagonal entries $-1, \ldots, -1$ ($d-1$ times) and $d, \ldots, d$ (d times). ■

Let us define

$$\Sigma_d := \left\{ (\gamma_0, \ldots, \gamma_d) \in \mathbb{C}^{d+1} \;\middle|\; \gamma_d \Delta_d(\gamma_0, \ldots, \gamma_d) = 0 \right\}.$$

The set Σ_d is a slight variant of the classical discriminant variety $\nabla_{\{0,\ldots,d\}} \subset \mathbb{P}^d_{\mathbb{C}}$ of degree d bivariate homogeneous polynomials with a degenerate root. We will thus be identifying the polynomial $\gamma_0 + \cdots + \gamma_d x^d \in \mathbb{C}[x]$ with the point $(\gamma_0, \ldots, \gamma_d) \in \mathbb{C}^{d+1}$. In the Appendix, we will prove the following useful algebraic result:

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Lemma 1.2 *Assume $d \geq 1$ and $\gamma_d \Delta_d(\gamma_0, \ldots, \gamma_d) \neq 0$ for some $(\gamma_0, \ldots, \gamma_d) \in \mathbb{C}^{d+1}$. Then any complex root of $\gamma_0 + \cdots + \gamma_d x^d$ has multiplicity* 1.

It will also be useful to recall the following basic fact, easily provable via polynomial division.

Proposition 1.3 *If $f \in \mathbb{C}[x]$ has degree $d \geq 1$ then f has at most d distinct complex roots.* ■

Proposition 1.3 is sometimes called the *Weak* Fundamental Theorem of Algebra.

We will also need some results on building paths avoiding complex hypersurfaces and bounding roots of polynomials:

Proposition 1.4 *(See HW#1!) If $g \in \mathbb{C}[x_1, \ldots, x_N] \setminus \{0\}$ has complex zero set Z then $\mathbb{C}^N \setminus Z$ is analytically path-connected: For any $p, q \in \mathbb{C}^N \setminus Z$ there is an analytic $\phi : [0,1] \longrightarrow \mathbb{C}^n \setminus Z$ with $\phi(0) = p$ and $\phi(1) = q$.* ■

Lemma 1.5 *If $f(x) = \gamma_0 + \cdots + \gamma_d x^d \in \mathbb{C}[x]$ is a univariate polynomial of degree $d \geq 1$ then all the complex roots of f lie in the open disk of radius $2 \max\limits_{i \in \{0,\ldots,d-1\}} \left|\frac{\gamma_i}{\gamma_d}\right|^{\frac{1}{d-i}}$.* ■

Lemma 1.5 is in fact a very special case of our estimate (from the first day of lecture) on complex root norms via slopes of lower edges of Archimedean Newton polygons (see [AKNR18, Thm. 1.7]). In particular, the radius above is simply a disguised form of e^s where s is the maximal slope of an edge of ArchNewt(f). As stated above, the bound dates back to work of Fujiwara (see [Fuj16] and [RS02, pp. 243–249], particularly Bound 8.1.11 on page 247 of the latter reference), and admits a short proof by bounding the norm of a suitable geometric series.

Finally, we'll need the following two analytic results.

The Extreme Value Theorem. *(See, e.g., [Mun00].) If $S \subset \mathbb{R}^N$ is closed and bounded and $f : S \longrightarrow \mathbb{R}$ is continuous, then there exist $a_1, a_2 \in S$ with $f(a_1) = \inf_{x \in S} f(x)$ and $f(a_2) = \sup_{x \in S} f(x)$. In particular, f is bounded.* ■

The Implicit Function Theorem. *(See, e.g., [Hub15].) If $G : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is an analytic function with $G(t_0, x_0) = 0$ and $\frac{\partial G}{\partial x}|_{(t_0,x_0)} \neq 0$ for some $(t_0, x_0) \in \mathbb{C}^2$, then there is an open disk $U \subset \mathbb{C}$ containing t_0, and an analytic function $\mathcal{X} : U \longrightarrow \mathbb{C}$ (called a* branch of the implicit function defined by $G(t,x) = 0$*) with $\mathcal{X}(t_0) = x_0$ and $G(t, \mathcal{X}(t)) = 0$ for all $t \in U$.* ■

2 The Proof of the Fundamental Theorem of Algebra

We first observe that the case $d = 1$ is trivial: $\gamma_0 + \gamma_1 x$ has only the root $-\gamma_0/\gamma_1$, which has multiplicity 1 since $f'(x)$ is identically γ_1, and $\gamma_1 \neq 0$ since we assumed the degree of f is $d = 1$. So we may assume $d \geq 2$ henceforth.

We then separate our proof into two cases: (1) $f \notin \Sigma_d$ and (2) $f \in \Sigma_d$. The first case is harder so we take care of it first.

2.1 The Case $f \notin \Sigma_d$

First observe that $x^d - 1 \notin \Sigma_d$ thanks to Proposition 1.1. Since Σ_d is the complex zero set of a polynomial, Proposition 1.4 tells us that there is an analytic function

$$\phi = (\phi_0, \ldots, \phi_d) : [0,1] \longrightarrow \mathbb{C}^{d+1} \setminus \Sigma_d$$

with $\phi(0) = x^d - 1$ and $\phi(1) = f$. (Since we identify polynomials and points, we could have also said $\phi(0) = (-1, 0, \ldots, 0, 1)$ and $\phi(1) = (\gamma_0, \ldots, \gamma_d)$.) Let $G : [0,1] \times \mathbb{C} \longrightarrow \mathbb{C}$ be the analytic function defined by $G(t,x) := \phi_0(t) + \cdots + \phi_d(t)x^d$, i.e., G simply evaluates the polynomial $\phi(t)$ on our path at the complex number x.

Our main trick then will be to use our path of polynomials to build d *paths of roots*. On one end, the root paths start at the $d^{\underline{\text{th}}}$ roots of unity. We will ultimately prove that, on the other end, our root paths yield d distinct complex roots for f. Since $f \notin \Sigma_d$ implies that each root of f has multiplicity 1, and Proposition 1.3 tells us that f can have no more than d complex roots, proving that f has d complex roots will complete our proof.

Letting $D \subseteq [0,1]$ denote the set of $t \in [0,1]$ such that $Z_t := \{\zeta \in \mathbb{C} \mid G(t,\zeta) = 0\}$ has cardinality d, it clearly suffices to prove $D = [0,1]$. Toward this end, we'll first establish the following facts:

(P_0) D is non-empty. (P_1) D is open. (P_2) D is closed.

To prove (P_0), first note that $x^d - 1$ has exactly d roots: They are $e^{2\pi\sqrt{-1}/d}, \ldots, e^{2\pi(d-1)\sqrt{-1}/d}$, thanks to Euler's 1748 formula for the exponential function. The Implicit Function Theorem applied to each of these roots then implies that there are a positive ε, and analytic functions $\mathcal{X}_1, \ldots, \mathcal{X}_d : [0,\varepsilon] \longrightarrow \mathbb{C}$, with $\{\mathcal{X}_1(0), \ldots, \mathcal{X}_d(0)\} = \left\{e^{2\pi\sqrt{-1}/d}, \ldots, e^{2\pi(d-1)\sqrt{-1}/d}\right\}$ and $G(t, \mathcal{X}_i(t)) = 0$ for all $t \in [0,\varepsilon)$ and $i \in \{1, \ldots, d\}$. Furthermore, since the $\mathcal{X}_i$ are continuous and $|\mathcal{X}_i(0) - \mathcal{X}_j(0)| > 0$ for all $i \neq j$, we can assume further that $|\mathcal{X}_i(t) - \mathcal{X}_j(t)| > 0$ for all $i \neq j$ and $t \in [0, \varepsilon')$, for some positive $\varepsilon' \leq \varepsilon$. Since Z_t has cardinality $\leq d$ for all t (thanks to Proposition 1.3), we thus obtain that $D \supseteq [0, \varepsilon')$. So (P_0) is proved, and we refer to continuous functions $\mathcal{X}_i$ satisfying $G(t, \mathcal{X}_i(t))$ for t in some open interval as *root paths*.

The proof of Statement (P_1) follows the proof of Statement (P_0) almost identically: If Z_t has cardinality d for some $t \in [0,1]$, then the Implicit Function Theorem applied to each of the $\zeta \in Z_t$ yields d root paths defined in an open neighborhood $U \ni t$. By continuity again, we obtain that there is an open set $U' \subseteq U$, with $U' \ni t$ as well, such that the root paths do not intersect at any $t \in U'$. So then $U' \subseteq D$ and (P_1) is proved.

To prove Statement (P_2) we first need to observe that $|\phi_0|, \ldots, |\phi_d|$ are continuous functions on the compact set $[0,1]$. So by the Extreme Value Theorem, there is a positive M with $|\phi_0(t)|, \ldots, |\phi_{d-1}(t)| \leq M$ for all $t \in [0,1]$. Furthermore, ϕ_d is nonzero on all of $[0,1]$ (by the definition of Σ_d) and thus $|\phi_d|$ is a positive on $[0,1]$. So by the Extreme Value Theorem once again, $|\phi_d|$ has a positive minimum m on $[0,1]$, and thus $\left|\frac{\phi_i(t)}{\phi_d(t)}\right| \leq M/m$ for all $i \in \{0, \ldots, d-1\}$ and $t \in [0,1]$. By Lemma 1.5 we thus obtain that

$$\mathcal{I} := \{(t,\zeta) \in [0,1] \times \mathbb{C} \mid G(t,\zeta) = 0\}$$

is a bounded set.

Suppose now that $t' \in [0,1] \setminus D$ is a limit point of D. Then there are sequences

$$\begin{gathered} \zeta_1^{(1)}, \zeta_2^{(1)}, \ldots \in \mathbb{C} \\ \vdots \\ \zeta_1^{(d)}, \zeta_2^{(d)}, \ldots \in \mathbb{C} \end{gathered}$$

such that $\left\{\zeta_i^{(1)},\ldots,\zeta_i^{(d)}\right\}$ has cardinality d for all $i\in\mathbb{N}$, $G\left(t_i,\zeta_i^{(j)}\right)=0$ for all $(i,j)\in\mathbb{N}\times\{1,\ldots,d\}$, and $\lim\limits_{i\to\infty} t_i=t'$.

We claim that the sequences of roots $\left(\zeta_i^{(1)}\right)_{i\in\mathbb{N}},\ldots,\left(\zeta_i^{(d)}\right)_{i\in\mathbb{N}}$ always remain at a positive distance from each other. More precisely, we claim that there is a positive δ with $\left|\zeta_i^{(j)}-\zeta_i^{(j')}\right|\geq\delta$ for all $j\neq j'$ and $i\in\mathbb{N}$. For if not, then there is a pair (j,j') with $j\neq j'$ and a subsequence $i_1,i_2,\ldots$ with $\lim_{n\to\infty}\left|\zeta_{i_n}^{(j)}-\zeta_{i_n}^{(j')}\right|=0$, i.e., both $\zeta_{i_n}^{(j)}$ and $\zeta_{i_n}^{(j')}$ tend to a common limit ζ. Since G and $\frac{\partial G}{\partial x}$ are continuous, we would then obtain $G(t',\zeta)=0$ and

$$\left.\frac{dG}{dx}\right|_{(t',\zeta)}=\lim_{n\to\infty}\frac{G\left(t_{i_n},\zeta_{i_n}^{(j)}\right)-G\left(t_{i_n},\zeta_{i_n}^{(j')}\right)}{\zeta_{i_n}^{(j)}-\zeta_{i_n}^{(j')}}=\lim_{n\to\infty}\frac{0-0}{\zeta_{i_n}^{(j)}-\zeta_{i_n}^{(j')}}=0.$$

But then, this contradicts the fact that $\phi([0,1])$ avoids Σ_d. So the sequences $\left(\zeta_i^{(1)}\right)_{i\in\mathbb{N}},\ldots,\left(\zeta_i^{(d)}\right)_{i\in\mathbb{N}}$ are indeed separated by some positive δ.

By this separation, and from the fact that $\mathcal{I}$ is closed and bounded, we then obtain that each sequence $\left(\zeta_i^{(j)}\right)_{i\in\mathbb{N}}$ has a limit point $\zeta^{(j)}$, and these limit points are all separated by a distance at least δ from each other. In particular, again using the continuity of G, we obtain $G\left(t',\zeta^{(j)}\right)=0$ for all $j\in\{1,\ldots,d\}$. So $Z_{t'}$ has cardinality $\geq d$ and thus, by Proposition 1.3 again, $t'\in D$. So we at last have Statement (P_2).

To conclude, consider the $\omega:=\inf\limits_{t\in[0,1]\setminus D} t$. Then $\omega\in D$ is impossible, since D is open by (P_1). Likewise, $\omega\in[0,1]\setminus D$ is impossible, since then ω would be a limit point of D (thanks to (P_0)), and (P_2) would imply that $\omega\in D$. So we must in fact have $D=[0,1]$ and we are done. ■

2.2 The Case $f\in\Sigma_d$

First note that we may assume $\gamma_d\neq 0$, for otherwise we could simply decrease d and reduce to a lower degree case.

Now let $L(t):=(1-t)(\gamma_d x^d-\gamma_d)+tf(x)$ and observe that the coefficient of x^d in $L(t)$ is exactly γ_d for all $t\in\mathbb{C}$. So the roots of $L(0)$ are the roots of unity, $L(1)=f(x)$, and $L(t)$ has degree d in x for all $t\in[0,1]$. Furthermore, since $x^d-1\notin\Sigma_d$ (as already observed in our last case), $\Delta_d(L(t))$ is a polynomial of positive degree $\leq 2d-1$. So, by restricting L to $[\varepsilon,1]$ for some suitable $\varepsilon\geq 0$ and rescaling t, we can build an analytic path $\psi=(\psi_0,\ldots,\psi_d):[0,1]\longrightarrow\mathbb{C}^{d+1}$ with $\psi(0)=g$ for some $g\in\mathbb{C}^{d+1}\setminus\Sigma_d$ and *only* intersecting Σ_d at $\psi(1)=f$. We will use ψ to build root paths that prove that f has at least one root in $\mathbb{C}$ and then conclude by long division.

More precisely, similar to our last case, let

$$H(t,x):=\psi_0(t)+\cdots+\psi_d(t)\zeta^d$$
$$W_t:=\{\zeta\in\mathbb{C}\mid H(t,\zeta)=0\}$$
$$\mathcal{J}:=\{(t,\zeta)\in[0,1]\times\mathbb{C}\mid H(t,\zeta)=0\}.$$

Thanks to the framework of the $f\notin\Sigma_d$ case (which we've already proved), it is clear that H

is analytic, $\mathcal{J} \setminus \Sigma_d$ is a disjoint union of d real analytic curves, and $\mathcal{J}$ is closed and bounded. In particular, these curves induce (possibly overlapping) limit points $(1, \rho_1), \ldots, (1, \rho_d)$. Furthermore, since H is continuous, f must vanish at each ρ_i. So f has at least one root $\rho \in \mathbb{C}$ and, letting μ (≥ 1) denote the multiplicity of ρ, it suffices to prove that $\frac{f(x)}{(x-\rho)^\mu}$ has exactly $d - \mu$ roots in $\mathbb{C}$ counting multiplicities. So by induction, we are done. ■

3 Consequences for Intersection Multiplicity and Discriminants

We first point out that the preceding proof yields a *geometric* definition of intersection multiplicity: For a degenerate root $\rho \in \mathbb{C}$ of f, following the notation of the last section, we can define the intersection multiplicity of a root ρ to be the number of paths of $\mathcal{J}$ ending at ρ, i.e., the number of ρ_i equal to ρ. With just a little more work (see, e.g., [BCSS98, Ch. 10]), we can show that this definition makes sense, i.e., our definition is independent of how we build a path in $\mathbb{C}^{d+1} \setminus \Sigma_d$ tending to f. It turns out that this definition of intersection multiplicity generalizes easily to higher-dimensions, e.g., Bézout's Theorem (see [BCSS98, Ch. 10] for the details).

A useful consequence of the Fundamental Theorem of Algebra is the following inverse of the implication from Lemma 1.2:

Lemma 3.1 *Assume $d \geq 2$, $f(x) = \gamma_0 + \cdots + \gamma_d x^d$, and $\Delta_d(\gamma_0, \ldots, \gamma_d) = 0$ for some $(\gamma_0, \ldots, \gamma_d) \in \mathbb{C}^{d+1}$ with $\gamma_d \neq 0$. Then f has a degenerate root, i.e., there is a $\zeta \in \mathbb{C}$ with $f(\zeta) = f'(\zeta) = 0$.*

We prove Lemma 3.1 in the Appendix.

4 A Root Approximation Algorithm from our Proof

Given any $f \in \mathbb{C}[x]$, consider the function $H(t, x) := (1-t)(x^d - 1) + tf(x)$. Let N be a large integer to be specified later, and consider the following iteration: Define $z_0^{(1)}, \ldots, z_0^{(d)}$ to be the $d^{\underline{\text{th}}}$ roots of unity, and then set

$$z_{n+1}^{(j)} := z_n^{(j)} - \left(\left. \frac{\partial H}{\partial x} \right|_{\left(\frac{n+1}{N}, z_n^{(j)}\right)} \right)^{-1} H\left(\frac{n+1}{N}, z_n^{(j)}\right)$$

for all $j \in \{1, \ldots, d\}$ and $n \in \{0, \ldots, N-1\}$. This is the most basic *homotopy continuation method* for approximating the roots of f: Under certain reasonably enforceable conditions, $z_N^{(1)}, \ldots, z_N^{(d)}$ are high-precision approximations to all the roots of f.

The key trick behind this method is that the sequences $\left(z_n^{(1)}\right)_{n \in \mathbb{N}}, \ldots, \left(z_n^{(d)}\right)_{n \in \mathbb{N}}$ yield a discrete approximation to a piece of the *incidence variety*

$$\mathcal{J} := \left\{(c_0, \ldots, c_d, \zeta) \in \mathbb{C}^{d+1} \times \mathbb{C} \mid c_0 + \cdots + c_d \zeta^d = 0\right\}.$$

In particular, the incidence variety consists of all pairs consisting of a polynomial and its roots, and we are trying to recover a point on $\mathcal{J}$ from its projection onto the first factor:

the coefficient space $\mathbb{C}^{d+1}$. The preceding iteration takes d (easily obtainable) points on $\mathcal{J}$ and then applies Newton's method (to deformations of f) to follow the d curves leading to the (hard to find) pairs $(f,\zeta_1),\ldots,(f,\zeta_d)$ where $\left\{\zeta^{(1)},\ldots,\zeta^{(d)}\right\}$ is the multi-set of roots of f (appearing with appropriate multiplicity).

If by luck we have that the line segment $\mathcal{L} := \left\{(1-t)(x^d-1)+tf(x)\right\}_{t\in[0,1]}$ does *not* intersect Σ_d, and N is sufficiently large, then one can in fact guarantee that the underlying Newton iterations are all well-defined and that the points $z_N^{(1)},\ldots,z_N^{(d)}$ are *approximate roots* of f. One should of course not rely on luck. However, the assumption that $\mathcal{L}$ does not intersect Σ_d actually holds with probability 1 for many natural probability measures on the space of coefficients $\mathbb{C}^{d+1}$. And, in the event that $\mathcal{L}$ intersects Σ_d, there are well-known strategies to modify the underlying path and iteration.

As for the size of the paramter N, there is a rigorous notion of *approximate root* (due to Smale [Sma81]) that gives a practical sufficient condition for an approximation $z_N^{(j)}$ to be efficiently refinable (to arbitrarily small distance from a true root) via Newton's method applied directly to $\left(f, z_N^{(j)}\right)$. We will make these notions precise later in lecture, but three seminal references on analyzing how large N should be are [SS94, BCSS98, **?**].

This approach to numerical solving has a transparent generalization to systems of multivariate polynomial equations (see, e.g., [SW05]), and has already been implemented and proven quite valuable. An excellent example is the package `Bertini`, which is freely downloadable after a simple google search. Homotopy methods thus possess a certain conceptual beauty, and making them faster and fully reliable leads to many active research problems in (numerical) algebraic geometry.

Appendix: Proving the Equivalence of Vanishing Discriminants and the Presence of Degenerate Roots

Collecting powers of x, it is easy to prove the following proposition, which we'll use below:

Proposition 4.1 *For any $d\geq 2$, and complex constants α_i, β_i, γ_i, and u_i, we have that the matrix product identity*

$$[\alpha_0,\ldots,\alpha_{d-2},\beta_0,\ldots,\beta_{d-1}]\mathrm{Syl}_d(f,f')=[u_0,\ldots,u_{2d-2}]$$

holds if and only if the polynomial identity

$$(\alpha_0+\cdots+\alpha_{d-2}x^{d-2})f(x)+(\beta_0+\cdots+\beta_{d-1}x^{d-1})f'(x)=u_0+\cdots+u_{2d-2}x^{2d-2}$$

holds. ■

The Proof of Lemma 1.2

If $d=1$ then $\gamma_1\Delta_1(\gamma_0,\gamma_1)=\gamma_1$ and $\gamma_1\neq 0$ clearly implies a unique root for $\gamma_0+\gamma_1 x$ of multiplicity 1. So let us assume $d\geq 2$. Then $\gamma_d\Delta_d(\gamma_0,\ldots,\gamma_d)\neq 0$ implies that $\det \mathrm{Syl}_d(f,f')\neq 0$, and thus the linear system

$$[r_0,\ldots,r_{d-2},s_0,\ldots,s_{d-1}]\mathrm{Syl}_d(f,f')=[1,0,\ldots,0]$$

has a unique solution.

By Proposition 4.1 we then have that

$$(r_0+\cdots+r_{d-2}x^{d-2})f(x)+(s_0+\cdots+s_{d-1}x^{d-1})f'(x)=1$$

identically. This implies, in particular, that f and f' can have no complex roots in common: Evaluating our last polynomial linear combination at such a root would imply that $0=1$. So we are done. ■

The Proof of Lemma 3.1

Under the stated hypotheses, $\det \mathrm{Syl}_d(f, f')=0$. So Proposition 4.1 implies that there must be polynomials $r, s \in \mathbb{C}[x]$, with degrees respectively bounded from above by $d-2$ and $d-1$, satisfying $rf + sf' = 0$ identically. In particular, the factorizations of rf and $-sf'$ must be identical. So by the Fundamental Theorem of Algebra (which we've at last proved), f has exactly d linear factors (counting multiplicity) since $\gamma_d \neq 0$. Since s has no more than $d-1$ linear factors, f and f' must have a common linear factor. So we are done. ■

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